Nonlinear Suboptimal Guidance with Impact Angle Constraint for Slow Moving Targets in 1-D Using MPSP

Priya G. Das and Radhakant Padhi
Department of Aerospace Engineering, Indian Institute of Science, Bangalore, India

ABSTRACT

Using a recently developed method named as model predictive static programming (MPSP), a nonlinear suboptimal guidance law for a constant speed missile against a slow moving target with impact angle constraint is proposed. In this paper MPSP technique leads to a closed form solution of the latax history update for the given problem. Guidance command is the latax, which is normal to the missile velocity and the terminal constraints are miss distance and impact angle. The new guidance law is validated by considering the nonlinear kinematics with both lag-free and first order autopilot delay.

Keywords- optimal guidance law, impact angle constraint, model predictive guidance, MPSP

I. INTRODUCTION

Guidance provides maneuvering commands to a flight vehicle to follow a desired path. A popular classical guidance law is the “proportional navigation (PN)” [10] (and its variants), which has been both well studied and successfully implemented in many flight vehicles. It is also well-known that the PN guidance law serves as an “optimal guidance law” [2]. However, this optimality results come with certain assumptions like linearized engagement geometry, proper choice of the navigation constant etc. Modern techniques, on the other hand, formulate the problem in the framework of optimal control theory and attempt to come up with more powerful guidance laws. For example, in a recent paper [7], the authors have come up with an optimal guidance law that not only minimizes the miss distance, but also leads to a desired impact angle (an impact angle constrained guidance law leads to enhancement of warhead effect for anti-ship or anti-tank missiles).

In reference [7] (as well as in many other literatures), the optimal guidance law available relies on linearized engagement geometry, which facilitates the usage of “linear optimal control theory” [2]. Linear optimal guidance laws, however, either have limited application domain or degraded performance (because of the linearized engagement geometry). Both to make the application domain wider as well as better performance, it is desirable that the nonlinear engagement kinematics is considered in developing the guidance law itself. A variation of nonlinear optimal control theory which is widely used in practice (especially in chemical engineering problems) is model predictive control [3], where the idea is to use output dynamics instead of state dynamics (to reduce numerical complexity). Here the output is predicted for a finite time window using a guess control and based on the error in the output the control history is updated by minimizing a performance index.

Guidance law using impact angle constraint which is already available in the literature uses a linearized dynamics. However in this paper the system dynamics are not linearized for latax calculation. Here we present a new model predictive static programming method which appeared in the literature recently for solving similar problems [4]. The technique leads to an approximate closed form solution for the given problem. The computational requirements of this method is much lesser and the algorithm can be implemented online.

The simulation results show that MPSP technique is quite promising to address the problem preserving the nonlinear kinematics of engagement. The method has the advantage that the latax calculation is carried out using the nonlinear dynamics. The guidance law is validated both for lag free and first order delay system. The simulation result of linearized system with first order delay shows a singularity at the end of guidance command, which is sometimes called as “blind zone”, since latax command is not realizable. The simulation result for first order delay system shows that Guidance command does not show a blind zone, when MPSP technique is used.

II. THE ENGAGEMENT DYNAMICS

Figure 1 shows the engagement geometry for a stationary and slow moving target. The missile is moving with a constant speed and the target under consideration is assumed to be moving at a very low speed along X direction. The guidance command is the lateral acceleration (latax), which is the acceleration normal to the velocity vector [8]. The equation of motion for this homing missile problem is formulated both for lag free and first order autopilot delay system.

A. Equation of motion for Lag free Autopilot

Impact angle, for a slow moving target is the angle between missile and target velocities at the time of interception [8]. The terms \( \alpha_m, V_m, \theta_m \) and \( \theta_{mf} \) are defined as the
acceleration applied normal to the velocity vector, the missile velocity, the flight path angle and the predetermined impact angle respectively. Then the equation of motion of lag -free autopilot is given by [8]

\[ \dot{z} = V_m(t) \sin \theta_m(t) \]  
\[ V_m(t) \dot{\theta}_m(t) = a_m(t) \]  (1)  (2)

System dynamics in normalized form can be written as

\[ \dot{z}_N = \frac{V_m \sin(\theta_m^* \theta_{mN})}{\sin \theta_{mN}} \]  (3)
\[ \dot{\theta}_{mN} = \frac{a_m}{V_m \theta_{mN}^*} \]

where \( z_N \triangleq z^* / z^* \) and \( \theta_{mN} \triangleq \theta_m / \theta_{mN}^* \) are the normalized state variables. The symbols \( z^* \) and \( \theta_{mN}^* \) represent the normalizing variables. The guidance law discussed in earlier paper [5], the target position is stationary and time-to-go \( (t_{go}) \) automatically take care of target position. So using suitable time-to-go estimation the work is extended to moving target in a general direction. The target moving along X-direction does not affect any state but only the time-to-go for extending the guidance law. Meanwhile the concept of “predicted intercept point (PIP)” is introduced for generating a virtual target for a missile engaging a target moving along X-axis (with a speed \( V_t \)). Considering PIP as virtual target the missile engages it and finally hits the target at desired impact angle. The position of PIP and the modified guidance for linearized system changes as given in Appendix VI.

B. Equation of motion for First order Autopilot

The dynamics of first order autopilot can be written as

\[ (a_m - a_c) + \frac{1}{V}(a_m - a_c) = 0 \]  (4)

The closed form solution of (4) is given as

\[ a_m = a_c - e^{-\frac{1}{V}}(a_{c0} - a_{m0}) \]  (5)

where \( a_m \) is the achieved missile latax and \( a_c \) is the commanded missile latax. So the latax for first order is replaced by the closed form solution given by (5). Now the system model can be written as [8]

\[ \dot{z} = V_m(t) \sin \theta_m(t) \]  (6)
\[ V_m(t) \dot{\theta}_m(t) = a_c - e^{-\frac{1}{V}}(a_{c0} - a_{m0}) \]  (7)

System dynamics in normalized form can be written as

\[ \dot{z}_N = \frac{V_m \sin(\theta_m^* \theta_{mN})}{\sin \theta_{mN}} \]  (8)
\[ \dot{\theta}_{mN} = \frac{a_c}{V_m \theta_{mN}^*} - e^{-\frac{1}{V}}(a_{c0} - a_{m0}) \]

where \( z_N \triangleq z^* / z^* \) and \( \theta_{mN} \triangleq \theta_m / \theta_{mN}^* \) are the normalized state variables. The target position is stationary and time-to-go \( (t_{go}) \) automatically take care of target position. So using suitable time-to-go estimation the work is extended to moving target in a general direction. The target moving along X-direction does not affect any state but only the time-to-go for extending the guidance law. Meanwhile the concept of “predicted intercept point (PIP)” is introduced for generating a virtual target for a missile engaging a target moving along X-axis (with a speed \( V_t \)). Considering PIP as virtual target the missile engages it and finally hits the target at desired impact angle. The position of PIP and the modified guidance for linearized system changes as given in Appendix VI.

C. Problem Objectives

Main objective of the problem is to design a nonlinear suboptimal missile guidance with terminal impact angle constraint using MPSP technique. MPSP technique leads to an approximate closed form solution of the latax history update preserving the nonlinear kinematics of engagement. Main goal of the problem is to achieve zero miss distance with impact angle constraint. The paper gives an approximate closed form suboptimal guidance law for both lag free and first order autopilot design. The mathematical formulation of the MPSP technique is explained in fair detail in the next section.

III. Nonlinear Guidance Law Using MPSP Technique

Consider a general nonlinear systems in discrete form, the state and output dynamics of which are given by [4]

\[ X_{k+1} = f_k(X_k, U_k) \]  (10)
\[ Y_k = h(X_k) \]  (11)

where \( X \in \mathbb{R}^n, U \in \mathbb{R}^m, Y \in \mathbb{R}^p \) and \( k = 1, 2, \ldots, N \) are the time steps. The primary objective is to come up with a suitable control history \( U_k, k = 1, 2, \ldots, N - 1 \), so that the output at the final time step \( Y_N \) goes to a desired value \( Y_N^* \), i.e. \( Y_N = Y_N^* \). In addition, we aim to achieve this task with minimum control effort , where \( k = 1, 2, \ldots, N \) are time steps. The steps involved in MPSP technique are summarized as follows. Start from a “guess history” of the control solution. With the application of such a guess history the objective is not met and hence there is a need to improve the solution. So we present a way to compute an error history of control variable which needs to be subtracted from previous history to get an improved control history. Iteration continues until the objective is met. To meet the objective \( Y_N = Y_N^* \), define the error in the output as \( \Delta Y_N = Y_N - Y_N^* \). Note

2
that the technique presented here comes up with a control update history in closed form, and hence, the computational requirement is substantially lesser. Hence, algorithm can be used online. Next, we present the mathematical details of the MPSP design.

Expanding $Y_N$ about $Y_N^*$ using Taylor series expansion

$$Y_N = Y_N^* + \left[ \frac{\partial Y_N}{\partial X_N} \right] dX_N + HOT$$

(12)

where $HOT$ contains the higher order terms. From (12) we can write the error in the output as

$$Y_N - Y_N^* = \left[ \frac{\partial Y_N}{\partial X_N} \right] dX_N + HOT$$

(13)

Using small error approximation, we write

$$\Delta Y_N \approx dY_N = \left[ \frac{\partial Y_N}{\partial X_N} \right] dX_N$$

(14)

However from (10), we can write the error in state at time step $(k+1)$ as

$$dX_{k+1} = \left[ \frac{\partial F_k}{\partial X_k} \right] dX_k + \left[ \frac{\partial F_k}{\partial U_k} \right] dU_k$$

(15)

where $dX_k$ and $dU_k$ are the error of state and control at time step $k$ respectively. Expanding $dX_N$ as in (15) (for $k = N-1$) and substituting it in (14), we get

$$dY_N = \left[ \frac{\partial Y_N}{\partial X_N} \right] \left( \left[ \frac{\partial F_{N-1}}{\partial X_{N-1}} \right] dX_{N-1} + \left[ \frac{\partial F_{N-1}}{\partial U_{N-1}} \right] dU_{N-1} \right)$$

(16)

Continue the process until $k = 1$, So we can write

$$dY_N = A dX_1 + B_1 dU_1 + \ldots + B_{N-1} dU_{N-1}$$

(17)

where

$$A \triangleq \left[ \frac{\partial Y_N}{\partial X_N} \right] \left[ \frac{\partial F_{N-1}}{\partial X_{N-1}} \right] \ldots \left[ \frac{\partial F_1}{\partial X_1} \right]$$

$$B_k \triangleq \left[ \frac{\partial Y_N}{\partial X_N} \right] \left[ \frac{\partial F_{N-1}}{\partial X_{N-1}} \right] \ldots \left[ \frac{\partial F_{k+1}}{\partial X_{k+1}} \right] \left[ \frac{\partial F_k}{\partial U_k} \right]$$

(18)

for $k = 1, \ldots, N-1$. Since the initial condition is specified, there is no error in the first term; which means $dX_1 = 0$. With this (17) reduces to

$$dY_N = B_1 dU_1 + B_2 dU_2 + \ldots + B_{N-1} dU_{N-1} = \sum_{k=1}^{N-1} B_k dU_k$$

(19)

Note that while deriving (19), we have assumed that the control variable at each time steps to be independent of the previous values of states and/or control. Intuitive justification of this assumption comes from the fact it is a decision variable, and hence, independent decision can taken at any point of time.

At this point, we would like to point out that if one evaluates each of the $B_k, k = 1, \ldots, (N-1)$ as in (18), it will be a computationally intensive tasks (especially when $N$ is high). However, fortunately it is possible to compute them recursively. For doing this, first we define $B_{N-1}^0$ as follows

$$B_{N-1}^0 = \left[ \frac{\partial Y_N}{\partial X_N} \right]$$

(20)

Next we compute $B_k^0, k = (N-2), \ldots, 1$ as

$$B_k^0 = \left[ \frac{\partial F_{k+1}}{\partial X_{k+1}} \right]$$

(21)

Finally, $B_k, k = (N-2), \ldots, 1$ can be computed as

$$B_k = B_k^0 \left[ \frac{\partial F_k}{\partial U_k} \right]$$

(22)

Equations (20)-(22) provides a recursive way of computing $B_k, k = (N-1), (N-2), \ldots, 1$, which leads to saving of computational time enormously.

In equation (19), we have $(N-1)m$ unknowns and $p$ equations. Usually $p < (N-1)m$, and hence, it is an under-constrained system of equations. Hence there is a scope for meeting additional objectives. We take advantage of this opportunity and aim to minimize the following objective (cost) function

$$J = \frac{1}{2} \sum_{k=1}^{N-1} (U_k^0 - dU_k)^T R_k (U_k^0 - dU_k)$$

(23)

where $U_k^0, k = 1, \ldots, (N-1)$ is the previous control history solution and $dU_k$ is the corresponding error in the control history. The cost function in (23) needs to be minimized subjected to the constraint in (19), where $R_k > 0$ (a positive definite matrix) is the weighting matrix, which needs to be chosen judiciously by the control designer. The selection of such a cost function is motivated from the fact that we are interested in finding a $l_2$-norm minimizing control history [4], since $(U_k^0 - dU_k)$ is the updated control value at $k$ (see (31)).

Equations (19) and (23) formulate an appropriate constrained static optimization problem. Hence, using optimization theory [2],[4], the augmented cost function is given by

$$J = \frac{1}{2} \sum_{k=1}^{N-1} (U_k^0 - dU_k)^T R_k (U_k^0 - dU_k) + \lambda^T (dY_N - \sum_{k=1}^{N-1} B_k dU_k)$$

(24)

Then the necessary conditions of optimality are given by

$$\frac{\partial F_k}{\partial dU_k} = -R_k (U_k^0 - dU_k) - B_k^0 \lambda = 0$$

(25)

$$\frac{\partial F_k}{\partial \lambda} = dy_N - \sum_{k=1}^{N-1} B_k dU_k = 0$$

(26)

Solving for $dU_k$ from (25), we get

$$dU_k = R_k^{-1} B_k^0 \lambda + U_k^0$$

(27)

Substituting for $dU_k$ from (27) into (26), it leads to

$$-A_k \lambda + b_k = dY_N$$

(28)
\[ A_{\lambda} = \begin{bmatrix} -1 & B_k \end{bmatrix}, \quad b_{\lambda} = \begin{bmatrix} -1 \end{bmatrix} \]

Note that \( A_{\lambda} \) is a \( p \times p \) matrix and \( b_{\lambda} \) is a \( p \times 1 \) vector. Assuming \( A_{\lambda} \) to be nonsingular, the solution for \( \lambda \) from (28) is given by

\[ \lambda = -A_{\lambda}^{-1}(dY_N - b_{\lambda}) \quad (29) \]

Using (29) in (27), it leads to

\[ dU_k = -B_k A_{\lambda}^{-1}(dY_N - b_{\lambda}) + U_k^0 \quad (30) \]

Hence, the updated control at time step \( k = 1, 2, \ldots, (N - 1) \) is given by

\[ U_k = U_k^0 - dU_k = R_k^{-1}B_k A_{\lambda}^{-1}(dY_N - b_{\lambda}) \quad (31) \]

One can refer [4] for more details regarding the MPSP technique. At this point, we would like to point out that we have used “small error approximation” in deriving the closed form control update. This approximation may not hold good in general. Hence the process needs to be repeated in an iterative manner before one arrives at the converged (optimal) solution, which is defined as the solution when \( Y_N \rightarrow Y_N^* \). Note that to minimize computational time, one iteration may be carried out at each instant of time following the principle of “iteration unfolding” [6]. The Problem specific equations for lag free system are obtained making use of the system dynamics (3). It can be discretized using Euler integration [1] as

\[ z_N(k + 1) = z_N(k) + \frac{\Delta \bar{v}_m \sin(\theta_{m} \theta_{m})}{\bar{z}} \quad (32) \]

\[ \theta_{m}(k + 1) = \theta_{m}(k) + \frac{\Delta a_m}{V_m \theta_{m}} \quad (33) \]

The required expressions to compute \( \partial F_k/\partial X_k \) from (15) are

\[ \frac{\partial F_k}{\partial X_k} = \begin{bmatrix} 1 & \frac{\Delta a_m}{V_m \theta_{m}} \cos(\theta_{m} \theta_{m}) \ 
0 & 1 \end{bmatrix} \quad \frac{\partial F_k}{\partial U_k} = \begin{bmatrix} 0 \
0 \end{bmatrix} \quad (34) \]

The Problem specific equations for a system with first order delay are obtained making use of the system dynamics (8). It can be discretized using Euler integration [1]

\[ z_N(k + 1) = z_N(k) + \frac{\Delta \bar{v}_m \sin(\theta_{m} \theta_{m})}{\bar{z}} \quad (32) \]

\[ \theta_{m}(k + 1) = \theta_{m}(k) + \Delta \left( \frac{a_m}{V_m} - e^{e^{Z}}(a_{m0} - a_{m0}) \right) \quad (33) \]

The required expressions to compute \( \partial F_k/\partial X_k \) and \( \partial F_k/\partial U_k \) are written as

\[ \frac{\partial F_k}{\partial X_k} = \begin{bmatrix} 1 & \frac{\Delta a_m}{V_m \theta_{m}} \cos(\theta_{m} \theta_{m}) \ 
0 & 1 \end{bmatrix} \quad \frac{\partial F_k}{\partial U_k} = \begin{bmatrix} 0 \
0 \end{bmatrix} \quad (34) \]

The main goal of the problem is to make \( Y_N \rightarrow Y_N^* \). For this particular problem \( Y_N = \begin{bmatrix} z \ 
\theta_{m} \end{bmatrix} \) and \( Y_N^* = \begin{bmatrix} 0 \ 
\theta_{m0} \end{bmatrix} \)

### IV. SIMULATION RESULTS

#### A. Numerical values of Parameters

The missile velocity is taken as a constant value of 300m/sec and the impact angle varies from –80 to 80. The target velocity is assumed as 50m/sec and moving along the X-direction. The various initial values of states used for simulation are tabulated as in Table 1.

The simulation results for both lag free and first order delay system are presented here. The result is being compared with linearized method. The basic idea regarding linearized method for stationary target, which is available in recent literature [7]. Review of linearized method is given as an appendix at the end of the paper for completeness. The idea is extended for a moving target with the introduction of predicted interception point. The guidance law is obtained using MPSP technique, which gives a closed form control update of the given nonlinear problem. The simulation results in Fig.2 and Fig.3 represent the comparison of optimal trajectories and latat between linearized method and MPSP approach for different impact angle for a lag free system. The trajectories are drawn for different impact angles such as –30°, –60°, –90° respectively. It is evident from the plots that the singularity effect issue at the end of guidance command is taken care of using MPSP technique. Positive negative turning effect also reduced using this new technique. The figure also shows the target trajectory which moves along X axis starting from the target initial X position and the interception takes place exactly at PIP point as was expected with the desired impact angle.

The simulation is repeated for different ranges, for example 4km, 5km, 6km, and 7km. The Figure.4 shows the responses of states and engagement parameters for different ranges for a lag free system. The Fig.4(a) shows the x value reaches at \( x_{PIP} \) at final time for all ranges and the corresponding Fig.4 (d) shows that the interception takes place at the desired impact angle of –80° for all ranges with a very low miss distance as is shown in Fig.4(c).

Figure5 and Fig.6 shows the comparison of optimal trajectories and latat between linearized method and MPSP approach for different ranges for lag free system. All the trajectories are drawn for \( \theta_{m0} = 30° \) and \( \theta_{m} = –80° \). As range increases the initial guidance command required for both cases are less but the command is required for a longer time to meet the expected range. The simulations are repeated for first order delay system and the it shows satisfactory performance for this case also. Figure7 and Fig.8 represents the comparison of optimal trajectories and latat between linearized method and MPSP approach for different

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Parameter Name</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_{m0} )</td>
<td>Missile position</td>
<td>0m 0m</td>
</tr>
<tr>
<td>( \gamma_{0} )</td>
<td>Target Position</td>
<td>4000 m 0km</td>
</tr>
<tr>
<td>( \phi_{m0} )</td>
<td>Initial heading angle</td>
<td>30°</td>
</tr>
</tbody>
</table>
impact angle for first order delay system for a range of 5km. The trajectories are generated for impact angle of $-10^0, -20^0, -40^0$ respectively. The figures are generated for different ranges also as like lag free system.
V. CONCLUSIONS

The paper discusses about the nonlinear optimal guidance law using MPSP technique, which gives a suboptimal closed form solution of the latax history update of the given problem. The paper discusses about a moving target along X-axis however the basic philosophy presented here can possibly be extended to moving target in all directions. The technique leads to a rapid update of guidance history, and hence can be implemented online. Optimal guidance law using linearized dynamics is already present in the literature, where the system dynamics is linearized and the latax is obtained using the linearized dynamics. The guidance laws is tested by assuming lag free and first order autopilots. In the design the guidance law is able to hit the target at predicted interception point at any desired range. It is important to note that the approach presented leads to an approximate closed form sub optimal solution of the guidance history update for the given nonlinear problem. Hence, the computational
requirements are low and it can be implemented online. The technique does not demand very high latax, and it is able to meet the required criteria of minimum miss distance.

REFERENCES


VI. APPENDIX - REVIEW OF LINEARIZED METHOD

The linearized method proposes an optimal guidance law with impact angle constraint with linearized dynamics [7]-[8]. It gives a closed form solutions for controlling the impact angle as well as terminal miss distance. It also proposes a generalized formulation of energy minimization optimal guidance law for constant speed missiles with an arbitrary system order for controlling both impact angle and miss distance. Optimal guidance law for lag free system and first order delay system are derived in this paper for completeness.

A. Guidance Law for Lag free System

System dynamics given by (1) can be linearized as [8]

\[
\dot{x} = V_m(t) \dot{\theta}_m(t) \\
V_m(t) \ddot{\theta}_m(t) = -a_m(t)
\]

(35)

(36)

The problem is to find \( u(t) \) which minimizes \( J \) defined by

\[
J = \frac{1}{2} \int_0^T \left( x(t_f) - x_f \right)^T S_f \left( x(t_f) - x_f \right) + \frac{1}{2} \int_0^T (u^T(\tau)R_u(\tau)d\tau
\]

subject to

\[
\dot{x} = Ax + Bu, \ x(0) = x_0
\]

(37)

(38)

The solution for this optimal control problem given by

\[
u^*(t) = \frac{V_m}{t_g} \left[ -6 \theta(t) + 4 \dot{\theta}_m(t) + 2 \theta_{mf} \right]
\]

(39)

When the target moves along reference with a speed \( v_t \), the solution of optimal control problem is rewritten in terms of predicted interception point (PIP) as

\[
x_{PIP} = x(t) + v_t t_g
\]

(40)

\[
u = -\left[ \frac{6V_m \theta_{PIP}}{t_g} + \frac{4x_2(t)}{t_g} \right]
\]

(41)

B. Guidance Law for System with First Order Delay

System dynamics for first order delay system is given by [8]

\[
\dot{\theta} = V_m(t) \sin(\theta_m(t))
\]

(42)

\[
V_m(t) \dot{\theta}_m(t) = -a_m(t)
\]

(43)

\[
\ddot{a}_m = \frac{u - a_c}{\tau}
\]

(44)

Optimal guidance law in this case is given as

\[
u = -\left[ V_m [-t_g w_1 z + w_2 \dot{\theta}_m + w_3 \theta_{mf}] + w_4 a_m \right]
\]

(45)

The following definitions are used in the optimal guidance law formulation for a first order delay system

\[
w_1 \triangleq \frac{1}{1/\Delta} (s_1 K_4 D_1 - V_m S_2 K_3 D_2)
\]

\[
w_2 \triangleq \frac{1/\Delta}{(s_1 D_1 (t_{go} K_4 - K_2))} + V_m S_2 D_2 (K_1 - t_{go} K_3)
\]

\[
w_3 \triangleq \frac{1/\Delta}{(s_1 K_2 D_1 - V_m S_2 K_3 D_2)
\]

\[
w_4 \triangleq \frac{1/\Delta}{(s_1 D_1 (K_2 D_2 - K_4 D_1) + V_m S_2 D_2 (K_3 D_1 - K_1 D_2))
\]

\[
\Delta \triangleq K_1 K_4 - K_2 K_3
\]

\[
K_1 \triangleq 1 + s_1 (1/2 \alpha^2) + \frac{t_{go}}{\alpha^2} - \frac{t_{go}^2}{\alpha} + \frac{t_{go}^3}{3} - 2 \frac{t_{go} e^{-\alpha t_{go}}}{\alpha^2} - \frac{e^{-2 \alpha t_{go}}}{2 \alpha^2}
\]

\[
K_2 \triangleq V_m S_2 (1/2 \alpha^2 - t_{go}/\alpha + \frac{t_{go}^2}{\alpha^2} + \frac{t_{go}^3}{3} - 2 \frac{t_{go} e^{-2 \alpha t_{go}}}{\alpha^2})
\]

\[
K_3 \triangleq s_1 K_2 / V_m S_2
\]

\[
K_4 \triangleq 1 + V_m S_2 (t_{go} + 2 e^{-\alpha t_{go}}/\alpha - 3/2 \alpha - e^{-2 \alpha t_{go}}/2 \alpha)
\]

\[
D_1 \triangleq \frac{t_{go}}{1/\alpha} - e^{-\alpha t_{go}}/\alpha
\]

\[
D_2 \triangleq \frac{(1/\alpha)}{V_m (1 - e^{-\alpha t_{go}})}
\]

\[
\alpha \triangleq \frac{1}{\tau}
\]

Note that for the first order lag system the guidance command goes to zero as \( t_{go} \rightarrow 0 \). When the target moves along reference with a speed \( v_t \), the optimal guidance law changes as

\[
u = -\left[ V_m [-t_g w_1 \theta_{PIP} + w_2 \dot{\theta}_m + w_3 \theta_{mf}] + w_4 a_m \right]
\]

(46)

One can refer [8] for more details. In this guidance command \( \theta_{PIP} \) is obtained from \( x_{PIP} \) using \( v_t \) information.